Strongly Lifting Modules

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Abstract:

In this paper we introduce the new concept of lifting modules, namely strongly lifting modules. Various properties and characterizations of strongly lifting modules are established and the relation between this type of module and some other known modules are discussed.

Keywords: Small submodule, Supplement submodule, Coclosed submodule, Lifting module, and Projective cover.

Introduction:

Throughout this paper all modules are assumed to be unital left modules over rings with identity, unless otherwise stated. Recall that:

1. A submodule A of an R-module M is called *small or superfluous* in M, denoted by $A \ll M$, if and only if for all submodule U of M, $A+U=M$, implies that $U=M$, [1].
2. An R- module M is called a *lifting module* if for all submodule N of M, there exist submodules A, and B in M such that $M=A\oplus B$, A submodule of N and N intersect B small in B (equivalently small in M), see [2,3].
3. A submodule N of a module M is called *coclosed* in M if $N/K \ll M/K$ implies that $N=K$ for all submodule K of M contained in N. It is easy to show that if N is a coclosed submodule in M, then for all X submodule of N, if X is a small submodule in M implies that X is a small submodule in N, see [3].
4. Let P be a module then an epimorphism $f: P\to M$ is called a *projective cover* of a module M if and only if $f: P\to M$ is a small epimorphism, and P is a projective, equivalently if and only if f is an epimorphism , P is a projective and $\ker f \ll P$, see [1,4,5].
5. An R- module M is called *semisimple* if and only if every submodule of M is a direct summand of M, see [1].
6. Let N and L be two submodules of a module M, then we say N is a *supplement* of L in M if and only if $M=N+L$, and $N\cap L \ll N$, see [5].
7. An R-module is *indecomposable* if it is not the direct sum of two non-zero submodules, see [5].
8. An R- module M is called *hollow module* if and only if every proper submodule of M is small in M, see [5].

Moreover lifting module, Generalized lifting module, and strongly FI-lifting have been studied by many authors (see [2,6, 7]), and this idea leads us to introduce the following concept:

§1 strongly lifting module:

In this section we introduce the new definition namely strongly lifting module and we will prove that every strongly lifting module is lifting module but the converse is not true in general, for this purpose we give an example, moreover in this section we give a necessary condition to become lifting module a strongly lifting. Also we prove that the direct summand of a strongly lifting modules is strongly lifting module.

Definition 1.1: An R-module M is called a strongly lifting R-module, if for each submodule N of M, N is either summand or small.

Proposition 1.2: Every strongly lifting R-module M is a lifting R-module.

Proof: for all N submodule of an R-module M, since M is a strongly lifting R-module so either $N \ll M$, or N is a summand of M. If $N \ll M$, then always there exists a trivial submodule 0 as a directed summand of M, that is $M = 0 \oplus M$ where 0 is in N and $N \cap M = N \ll M$, hence in this case we can say M is a lifting module. But if N is a direct summand of M, then there exists a submodule T in M such that $M = N \oplus T$, and clearly N is a submodule of itself and $N \cap T = 0$ which is always small in M. Thus M is a lifting module■

Note. The following example shows that a lifting module needs not be a strongly lifting module.

Example 1.3: Consider $M=Z_{12}$ as a Z-module, then it is clear that Z_{12} has only the following submodules:

$(\bar{0}) = \{\bar{0}\}$,

$(\bar{1}) = Z_{12} = M$,

$(\bar{2}) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$,

$(\bar{3}) = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$,

$(\bar{4}) = \{\bar{0}, \bar{4}, \bar{8}\}$,

$(\bar{6}) = \{\bar{0}, \bar{6}\}$.

Table 1: bellow shows that which two submodules of M span M.

+	$(\bar{0})$	$(\bar{1})$	$(\bar{2})$	$(\bar{3})$	$(\bar{4})$	$(\bar{6})$
$(\bar{0})$		M				
$(\bar{1})$	M	M	M	M	M	M
$(\bar{2})$		M		M		
$(\bar{3})$		M	M		M	
$(\bar{4})$		M		M		
$(\bar{6})$		M				

Table 2: bellow shows that which two submodules of M forms a direct summand of M.

$\oplus$	$(\bar{0})$	$(\bar{1})$	$(\bar{2})$	$(\bar{3})$	$(\bar{4})$	$(\bar{6})$
$(\bar{0})$		M				
$(\bar{1})$	M					
$(\bar{2})$						
$(\bar{3})$					M	
$(\bar{4})$				M		
$(\bar{6})$						

Now to show that M is a lifting module for all N submodule of M, we have to show that there exist submodules K, and K' in M such that $M = K \oplus K'$, K in N and $N \cap k' \ll M$. For $N = (\bar{0})$ there exist submodules $(\bar{0})$, and M in M such that $M = (\bar{0}) \oplus M$, $(\bar{0})$ in $(\bar{0})$, and $(\bar{0}) \cap M = (\bar{0}) \ll M$. For $N = (\bar{2})$ there exist submodules $(\bar{4})$, and $(\bar{3})$ in M such that $M = (\bar{3}) \oplus (\bar{4})$, (see table 2), $(\bar{4})$ in $(\bar{2})$, and $(\bar{2}) \cap (\bar{3}) = (\bar{6})$ which is small in M (see table 1). For $N = (\bar{3})$ there exist submodules $(\bar{4})$, and $(\bar{3})$ in M such that $M = (\bar{3}) \oplus (\bar{4})$, (see table 2), $(\bar{3})$ in $(\bar{3})$, and $(\bar{4}) \cap (\bar{3}) = (\bar{0}) \ll M$. For $N = (\bar{4})$ there exist submodules $(\bar{4})$, and $(\bar{3})$ in M such that $M = (\bar{3}) \oplus (\bar{4})$, (see table 2), $(\bar{4})$ in $(\bar{4})$, and $(\bar{4}) \cap (\bar{3}) = (\bar{0}) << M$. Finally for $N = (\bar{6})$ there exist submodules $(\bar{0})$, and M in M such that $M = (\bar{0}) \oplus M, (\bar{0})$ in $(\bar{6})$, and $(\bar{6}) \cap M = (\bar{6}) << M$.

But to see Z_{12} as Z-module is not strongly lifting Z-module we observe from the above mentioned table 2 that there exists a submodule namely $(\bar{2})$ which is not summand, and not small since from table 1 we have $(\bar{2}) + (\bar{3})$ =M, but $(\bar{3}) \neq M$.

Proposition 1.4: Every indecomposable lifting module is a strongly lifting module.

Proof. Let N be any submodule of M. If N is not a proper submodule of a lifting module M then clearly N becomes summand of M, but if N is a proper sbmodule of a lifting module M, so there exists submodules K, and K' such that $M = K \oplus K', K'$ submodule of N and $N \cap K << K$. But M is indecomposable, hence either $K = 0$, or $K' = 0$. If $K = 0$ then $K' = M$, but K' is a submodule of N and N is a submodule of M, thus $N = M$ which is contradiction with N is a proper submodule of M. Therefore $K' = 0$, and then $K = M$, so $N = N \cap M = N \cap K << K = M$, this means $N << M$. Therefore M is a strongly lifting module ■

Example 1.5: By the same way as in the previous example we can easily see that the set of all integer numbers Z, and the set of all rational numbers Q , over the ring Z are not strongly lifting modules. But every simple, and uniserial module (particularly $Z_{P\infty}$ as Z-module) are strongly lifting modules.

The following lemma gives some properties of small submodules which can be found in [1], and we need it later.

Lemma 1.6: Let K, L and N be submodules of M. Then:

(1) If $K << M$ and $f: M \to M'$ is a homomorphism then f(K)<<M'.
(2) If K in L, L in N, and $L << N$ then $K << M$.
(3) If $K << M$ and $L << M$, then $K + L << M$.
(4) If L is adirect summand of M and K in L with $K << M$, then $K << L$.

Proposition 1.7: Any direct summand of a strongly lifting module is a strongly lifting module.

Proof. Let M be a strongly lifting module, suppose that $M = M1 \oplus M2$, we want to show that M1 is a strongly lifting module, for this purpose let N be any submodule of M1, so N automatically becomes a submodule of a strongly lifting module M, hence $N << M$, or N summand of M, by lemma 1.5 if $N << M$ then N is a small in M1. Moreover if N summand of M so there exists a submodule T in M such that $M = N \oplus T$, we are done if we can show that N is a direct summand of M1. Now $M1 = M1 \cap M = M1 \cap (N \oplus T) = N \oplus (M1 \cap T)$ by modular law. Thus N is a direct summand of M1 ■

Remark 1.8: At this point one may ask the following question:

Is the converse of the above proposition true in general? The following example gives negative answer to this question: one can easily show that Z_2, and Z_8 as Z-modules are strongly lifting module, but $M = Z_2 \oplus Z_8$ is not.

§2 Properties of strongly lifting modules.

In this section we give some properties and characterizations of strongly lifting modules.

Proposition 2.1: Epimorphic image of a strongly lifting module is a strongly lifting module.

Proof. Let $f: M1 \to M2$ be a module epimorphism with M1 strongly lifting module. To show that M2 is also a strongly lifting module, let N be any submodule of M2, we must prove that N is summand of M2 or small in M2. We know that f^{-1} (N) is a submodule of M1, but M1 is a strongly lifting module so f^{-1} (N) is a summand or small in M1. If f^{-1} (N) is summand of M1 then there exists a submodule say B in M1 such that: $f^{-1}(N) \oplus B = M1$, hence $f(f^{-1}(N) \oplus B) = f(M1)$, but one can easily show that: $f(f^{-1}(N) \oplus B) = N \oplus f(B)$. Moreover we have f epimorphism hence: $N \oplus f(B) = M2$, which means that N is summand of M2. In other hand if f^{-1} (N) is small in M1, then by Lemma 1.6; $f(f^{-1}(N)) \ll f(M1)$, but this implies that $N \ll M2$ (since f is an epimorphism), hence M2 is a strongly lifting module■

Corollary 2.2: If M is a strongly lifting module, then so is M/N for all submodule N of M.

Proof. Since there exists a natural epimorphism $f: M \to M/N$, then by using above proposition the proof becomes clear■

Proposition 2.3: If f is an epimorhpism from any module P on to a strongly lifting module M and $ker f \ll P$, then P is also strongly lifting module.

Proof. For simplicity let $ker f = K$, then from the fundamental theorem of isomorphism we have P/K isomorphic to M, and hence P/K is a strongly lifting module. Now to prove P is strongly lifting module let N be any submodule of P, we must show that N is summand of P or small in P. Since P/K is a strongly lifting module, so $(N+K)/K$ is summand of P/K, or small in P/K. If $(N+K)/K$ is summand of P/K, then there exists a submodule say K'/K in P/K, where K is a submodule of K', and K' is a submodule of P, such that: $(P/K) = ((N+K)/K) \oplus (K'/K)$, but this implies that: $P = (N+K) \oplus K'$, by assumption $K = ker f \ll P$, so $P = N \oplus K'$ and this means that N is a summand of P. But if $(N+K)/K$ is small in P/K, then we claim that for all submodule U of P the equation N+U=P implies that $U = P$. We know $N + U = P$ implies that: $(N+U)/K = P/K$, Or $((N+K)/K) + ((U+K)/K) = P/K$, but $(N+K)/K$ is small in P/K, hence $(U+K)/K = P/K$, or equivalently $U + K = P$, but by assumption $K = ker f \ll$P. Thus $U = P$. therefore N is small in P■

Corollary 2.4: Let K be any small submodule of an R-module M, then M is a strongly lifting module if M/K is a strongly lifting module.

Proof . Is trivial■

Corollary 2.5: If f is an epimorhpism from a module P on to a module M and $ker f \ll P$, then P is strongly lifting if and only if M is a strongly lifting module.

Proof. Is clear (since we can deduce this proof from 2.1 and 2.3) ■

Proposition 2.6: Every semisimple module is a strongly lifting module.

Proof. Is clear■

The converse of propostion 2.6 is not true in general as we see in Z_4 as Z- module.

Note we denote by Rad(M) the radical of a module M.

Corollary 2.7:

(1) Every submodule of a semisimple module is a strongly lifting module.
(2) Every epimorphic image of a semisimple module is a strongly lifting module.
(3) The sum of semisimple modules is a strongly lifting module.
(4) If R is a semisimple ring then every R-module is a strongly lifting module.
(5) If every right and left R-module is injective, then R is a semisimple ring and M is a strongly lifting module.
(6) If every right and left R-module is projective, then R is a semisimple ring and M is a strongly lifting module.
(7) If every simple right R-module and every simple left R-module is projective, then R is a semisimple ring and M is a strongly lifting module.
(8) If every submodule of M has a supplement in M, and $Rad(M) = 0$ then M is a strongly lifting module.
(9) Let $f: P \to M$ be a projective cover of M, then P is a strongly lifting module if an only if M is a strongly lifting module.
(10) If M is artinian module, and $Rad(M) = 0$, then M is a strongly lifting module.
(11) If M is artinian, then M/Rad(M) is a strongly lifting module .

Proof. All the proofs are trivial■

The following corollary gives a necessary and sufficient condition for a strongly lifting module to be a semisimple module:

Corollary 2.8: Let M be a non- zero R-module which has unique small submodule, then M is a semisimple if and only if M is a strongly lifting module.

Proof. The proof is trivial■

Proposition 2.9: Every hollow module is a strongly lifting module

Proof. The proof is obvious■

The converse of the above proposition is not true in general as we see in Z_6 as Z-module.

Corollary 2.10:

(1) Every cyclic module which has unique maximal submodule is a strongly lifting module.
(2) Let M be a module, if every non-zero factor module of M is indecomposable then M is a strongly lifting module.
(3) Every local module is a strongly lifting module.
(4) If in a module M, we have Rad M, is a small and maximal, then M is a strongly lifting module.
(5) If P is a projective module, and indomorphism of P: End(P) is a local ring then P is a strongly lifting module.
(6) If P is a projective cover for a simple module then P is a strongly lifting module.
(7) If L is a supplement of a maximal submodule N in module M then L is a strongly lifting module.

Proof. Since each of the above cases gives a hollow module, so by proposition 2.9, the proof becomes clear■

Proposition 2.11: Every non-zero coclosed submodule of a strongly lifting module is also strongly lifting module.

Proof. Let N be a non-zero coclosed submodule of a strongly lifting module M. We must prove that N is a strongly lifting module for this purpose suppose that L is any submodule of N, then L is a submodule of M, but M is a strongly lifting module, hence $L \ll M$, or L summand of M. If $L \ll M$, and since N is coclosed in M, so $L \ll N$. Moreover, if L is a summand of M, then there exists L' in M such that $M = L \oplus L'$.Now $N = M \cap N = (L \oplus L') \cap N$, by modular law $N = L \oplus (L' \cap N)$, and this means that L is a summand of N, therefore N is a strongly lifting module■

Corollary 2.12: Every supplement submodule of a strongly lifting module is also a strongly lifting module.
Proof. The proof is obvious■

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پـــــوختـــــــه

لەم تویژینەوەیەدا ئیمە بیرۆکەی نوێمـــان خستۆتە ڕوو بۆمۆد یلی بەرزکەرەوە بەناوی مۆد یلی بەرزکەرەوەی بەهیز سیفات و جیا کەرەوەی جـیا جـیا مان دروست کردوە بۆ مۆدێلی بەرزکەرەوەی بەهێز، وە پەیوەندی نێوان ئەم جۆرە مۆدێلە و هە ندێك مۆدێلی تری ناسراو لێکۆلینەوەی لە سەرکراوە.

المقاس الرافع القوى

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الخلاصة

في هذا البحث، سوف نذكر مفهوما جديدا للمقاس (الموديول) الرافع تحت اسم المقاس الرافع القوى. لقد كوننا الخواص والمميزات مختلفة للمقاسات (الموديولات) الرافع القوي، وناقشنا العلاقة بين هذالنوع من الموديول و بعض الموديولات الاخرى المعروفة.